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1991 J. Phys. A: Math. Gen. 24 2053

(http://iopscience.iop.org/0305-4470/24/9/015)

J. Phys. A: Math. Gen. 24 (1991) 2053–2060. Printed in the UK

Linear decay in multilevel quantum systems

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Received 1 November 1990, in final form 11 January 1991

Abstract. We report a new decay phenomenon in multilevel quantum systems. Under certain conditions, there exist states of the system that decay rapidly at sharply localized times. This results in an average decay that is *linear* over most of the life of the excited state. Moreover, the total decay rate is not a monotonic function of the coupling strength between the excited and decayed states. We explore the mathematical basis of this phenomenon and consider its consequences for the quantum measurement problem.

Quantum mechanical decay has been found to be robustly exponential and, aside from memory effects, the principal theoretical departures involve very short and very long times†. We here report a remarkable new decay phenomenon which we term 'ticking,' in which particular initial states of a multilevel quantum system remain almost entirely undecayed for an extended period, subsequent to which they undergo a rapid transition to the decayed final state. The collection of all such 'particular initial states' substantially exhausts the Hilbert space of initial states; moreover, each of them has its own particular decay time and these times are evenly spaced. As a result, to a good approximation, the decay averaged over randomly selected initial excited states is linear. On casual inspection, the Hamiltonian giving rise to this effect shows no obvious tendency for repetitive or periodic behaviour, nor is the overall linear decay law dependent on a special selection of initial conditions.

Our interest in this phenomenon is threefold. First, there is the purely mathematical issue of how a rather simple linear dynamical system can give rise to sudden near discontinuities. Second, the average linear decay found here, a drastic departure from the generic exponential time dependence, appears to be a completely new phenomena in quantum mechanics. Third, this phenomenon was discovered while searching for the special measurement states whose existence is postulated in [2]. The states we have found not only provide examples of special states but come with the relative abundances required by that theory.

† The 'watched pot' short-time $O(t^2)$ dependence, long time power-law behaviour (and more) are discussed in Fonda *et al* [1]. Another comprehensive source is Newton [1]. There are other exotic phenomena, such as double poles, and these too are discussed by Newton. Recent interest, motivated by questions of proton decay, can be found in Gaemers and Visser [1]. The term 'watched kettle' was used in Peres [1].

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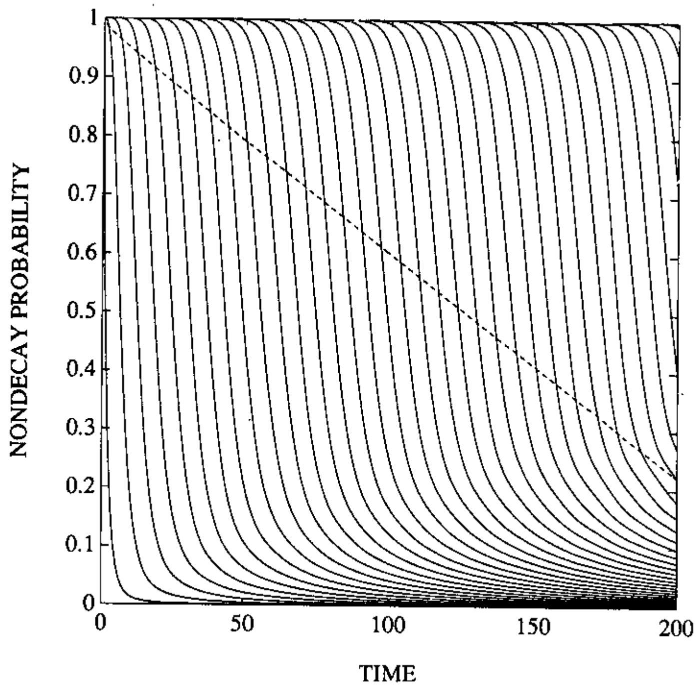

Figure 1. Non-decay probabilities for particular initial conditions. The broken line is the average. Space of decay products is infinite dimensional. The details of the Hamiltonian are given in the text preceding equation (8).

The phenomenon is illustrated in figure 1. What is shown is the decay (to a continuum, under an $M \to \infty$ limit of the Hamiltonian of equation (1), below) of a quantum system of 40 closely spaced levels. The broken line is the average of the probability that the system remains in any of the initial 40 states. The full curves, roughly speaking, are the decay histories of particular initial superpositions of the original 40 levels. (As we shall explain below, a more precise description is that for each time there is an initial state that attains the indicated degree of decay. Thus at time 100 there is a 13-dimensional subspace of states that have almost entirely decayed, a 22-dimensional subspace of states that have hardly decayed at all and five dimensions in transition.)

To derive the context within which figure 1 is calculated we consider the general Hamiltonian

$$H = \begin{pmatrix} H_N & C \\ C^\dagger & H_M \end{pmatrix}. \tag{1}$$

The Hilbert space $\mathcal{H}$ is $(N + M)$-dimensional, with the first N dimensions, the subspace $\mathcal{H}_N$, representing undecayed systems, and H_N being their Hamiltonian in the no-transition approximation. Correspondingly the decay products are an M-dimensional space $(\mathcal{H}_M)$ with Hamiltonian H_M. Finally C is the $N \times M$ matrix of transition amplitudes. This Hamiltonian is the multimode generalization of the Dicke Hamiltonian [3]. For radiative decay in infinite volume, decay products lie in a continuum, so that $M = \infty$, while in finite volume M is finite and may be quite small [4].

Let P project on $\mathcal{H}_N$. The initial state ψ_0 is in $\mathcal{H}_N$ and the amplitude for non-decay at time t is $\psi_{t,\text{non-decay}} = P\exp(-\mathrm{i}Ht)\psi_0$. It follows that the probability of non-decay can be written

$$\begin{aligned}\text{Prob(non-decay at time } t) &= \langle\psi_{t,\text{non-decay}}|\psi_{t,\text{non-decay}}\rangle \\ &= \langle\psi_0|A_t^\dagger A_t|\psi_0\rangle\end{aligned} \tag{2}$$

with $A_t = P\exp(-\mathrm{i}Ht)P$. For random initial conditions we average over $\mathcal{H}_N$ and the decay probability is $(1/N)\mathrm{Tr}A_t^\dagger A_t$. Ordinarily this is of the form $\mathrm{e}^{-\Gamma t}$ with $\Gamma \sim \mathrm{Tr}\,C^\dagger C$.

As indicated, we sought to determine whether there were special initial states for which there was significant departure from the average behaviour, in particular whether states could be found with either imperceptible decay or with decay that was imperceptibly different from the total. To this end we examined the spectrum of $A_t^\dagger A_t$ for a particular time t. If $A_t^\dagger A_t$ has an eigenvalue near one, the corresponding eigenstate represents an initial condition (in $\mathcal{H}_N$) for which, at time t, the operator $\exp(-\mathrm{i}tH)$ has hardly taken it out of the subspace $\mathcal{H}_N$. For a zero eigenvalue there is a state that has totally decayed.

We specialize to the case where the rows of C are smooth as functions of the $\mathcal{H}_M$ index, and in particular we will take them to be constant (although our results are insensitive to small variations). Such a situation obtains for atomic decay in the dipole approximation. For convenience we diagonalize H_N and H_M, which only rearranges C. The value taken by the elements of each row is picked randomly, although in most of the results we quote this value was of constant magnitude and uniformly distributed phase. In figure 2 we show the eigenvalues of $A_t^\dagger A_t$ as a function of time for $N = M = 40$, where the eigenvalues of H_N have been taken to be randomly distributed between 0 and 1 in accordance with the Wigner surmise [5] and the eigenvalues of H_M are uniformly distributed between -0.1 and 1.1.

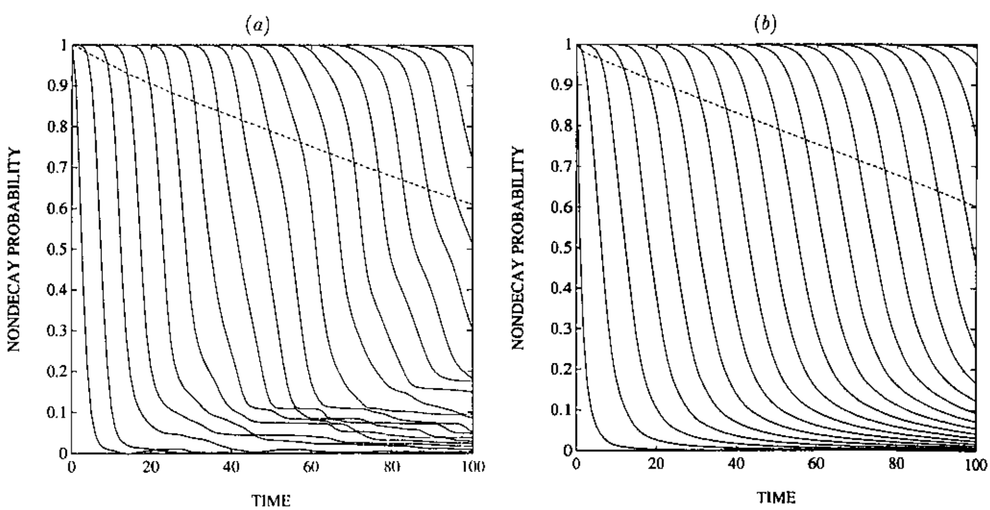

Figure 2. Non-decay probabilities for particular initial conditions. The broken line is the average. (a) Space of decay products is finite dimensional. (b) Space of decay products is infinite dimensional. (Same as figure 1, but for time period that matches that of figure 2(a)).

The ticking is evident in figure 2 and we next provide a theoretical explanation of this effect. First we dispose of $\mathcal{H}_M$ by going to an infinite-volume continuum limit.

For ψ we write $\binom{X}{Y}$ with X the restriction of ψ to $\mathcal{H}_N$. From $\mathrm{i}\dot{\psi} = H\psi$, it follows that

$$\dot{X} = -\mathrm{i}H_N X - C\int_0^t \mathrm{d}s \exp\left(-\mathrm{i}H_M(t-s)\right) C^\dagger X(s) \tag{3}$$

which is exact (for any C). For row-constant C we write $c_k = C_{kl}\sqrt{M}$. Equation (3) becomes

$$\dot{X} = -\mathrm{i}H_N X - c\int_0^t \mathrm{d}s\,(1/M)\,\mathrm{Tr}\,\exp\left(-\mathrm{i}H_M(t-s)\right) c^\dagger X(s). \tag{4}$$

In the limit $M \to \infty$, $(1/M)\mathrm{Tr}\,\mathrm{e}^{-\mathrm{i}H_M\tau}$ is $\pi\delta(\tau)$ and we have $\dot{X} = -\mathrm{i}H_N X - \pi cc^\dagger X$. An essential feature is already evident in that $cc^\dagger$ is proportional to a one-dimensional projection operator. We highlight this by defining $c = \sqrt{\alpha/\pi}\,\eta$ with η a unit vector and $\sqrt{\alpha} \geq 0$. Equation (4) becomes

$$\dot{X} = -\mathrm{i}H_N X - \alpha\eta\eta^\dagger X \tag{5}$$

or

$$\dot{Z} = -\alpha\eta(t)\eta(t)^\dagger Z \tag{6}$$

with $Z = \mathrm{e}^{\mathrm{i}H_N t}X$, and $\eta(t) = \mathrm{e}^{\mathrm{i}H_N t}\eta\mathrm{e}^{-\mathrm{i}H_N t}$. (We make the following remark. For $N = 1$, this development may be of pedagogical interest for presenting an explicit calculation of dissipation. With a minimum of algebra one can see how a continuum of final states, combined with the use of the initial condition $\|X(0)\|^2 = 1$, yield irreversible behaviour.) $X^\dagger X$ or $Z^\dagger Z$ represent the probability of non-decay. In equation (6) we see the beginnings of a qualitative explanation of the ticking phenomenon. Initially the time evolution consumes the component of X_0 $(= X(0))$ along η, but nothing else. For that one component (call it u) equation (6) has the form $\dot{u} = -\alpha u$, while the other components have zero derivative. Hence $A_t^\dagger A_t$ has one eigenvalue that is roughly $\mathrm{e}^{-2\alpha t}$ and other eigenvalues 1. Meanwhile $\eta(t)$ is rotating within $\mathcal{H}_N$ under the action of H_N. If α is large enough, $\langle\eta(0)|X(t)\rangle$ will become small before $\eta(t)$ has moved to a substantially different direction. Next, the new direction is consumed so that the decay is successively confined to one-dimensional subspaces. This explains both the existence of the ticking and its regularity, since the latter feature depends on the fact that $\eta(t)$ moves through $\mathcal{H}_N$ in a time homogeneous fashion with only a long time recurrence.

This argument is incomplete in that it suggests that increasing α should improve ticking, whereas numerically there is found to be an optimal value for sharp ticking which we have reason to believe is $1/\pi$ (for large N). In fact, a sufficient increase in α not only destroys ticking, it decreases *total* decay rate. This counter-intuitive property arises from a variant of the 'watched pot' effect [1]. To see this, consider equation (5) in the case of large α. We show that if $\langle\eta|X_0\rangle = 0$, then large α will prevent H_N from moving X_0 into the direction of η. Let Δt be such that $\langle\eta|H_N|X_0\rangle \ll 1/\Delta t \sim \alpha$ and consider the Trotter product [6] for the evolution

$$X(t) \sim \left[\exp\left(-\Delta t\alpha\eta\eta^\dagger\right)\exp\left(-\mathrm{i}H_N\Delta t\right)\right]^{t/\Delta t} X(0). \tag{7}$$

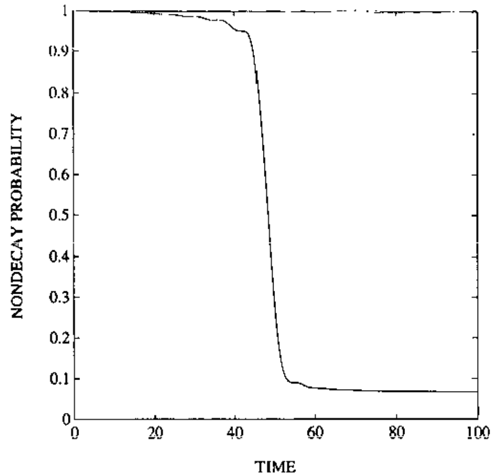

Figure 3. Time evolution of a Dirichlet wavepacket initially located at $\phi = -2\pi/5$.

A condition for the validity of equation (7) is $\alpha(\Delta t)^2\langle\eta|H_N|X_0\rangle \ll 1$. The operator $\exp(-\mathrm{i}H_N\Delta t)$ gives a term along η and a term orthogonal to it. The norm of the orthogonal term is unchanged by $\exp(-\Delta t\alpha\eta\eta^\dagger)$ while the piece along η is substantially wiped out (e.g. take $\Delta t = 5/\alpha$). Each of the $t/\Delta t$ iterations reduces the norm by $|\langle\eta|\exp(-\mathrm{i}H_N\Delta t)|X_0\rangle|^2$, or $\Delta t^2 \times \mathrm{O}(1)$. After $t/\Delta t$ iterations the norm is therefore $(1-(\Delta t)^2)^{t/\Delta t}$ or $\exp(-\Delta t)$. Thus the smaller we can take Δt, the less the change in norm.

Equation (5) is of the form $\dot{X} = -QX$ with $Q = \mathrm{i}H_N + \alpha\eta\eta^\dagger$. It follows that $A_t = \exp(-Qt)$ and $A_t^\dagger A_t = \mathrm{e}^{-Q^\dagger t}\mathrm{e}^{-Qt}$. This is a convenient form for analysis. For definiteness take $H_N = (1/2l)J_z$ and $(\eta)_m = 1/\sqrt{N}$, $m = 1,\ldots,N$, where $J_z = \mathrm{diag}(-l, -l+1, \ldots, l)$ and $N = 2l+1$. This is the system for which figure 1 was calculated, with the $N = 40$ and $\alpha = 1/\pi$. As a basis take the functions $\mathrm{e}^{\mathrm{i}m\phi}$. For X we write $X(t,\phi) = \sum X_m \mathrm{e}^{\mathrm{i}m\phi}$ so that H_N becomes $(1/2\mathrm{i}l)\partial/\partial\phi$. Equation (5) is then

$$\frac{\partial X(t,\phi)}{\partial t} = \frac{1}{2l}\frac{\partial X(t,\phi)}{\partial\phi} - \frac{\alpha}{2l+1}D(\phi)X(t,0) \tag{8}$$

with $D(\phi) \equiv \sin((2l+1)\phi/2)/\sin(\phi/2)$, the Dirichlet kernel. For large l, $D(\phi)$ is δ-function like and evolution under (8) can be described as follows. Take $X(0,\phi)$ to be a localized wavepacket with support away from $\phi = 0$. Initially it propagates with no change of shape or norm and the operator $\partial/\partial t - (1/2l)\partial/\partial\phi$ gives the packet a velocity $1/2l$. When this packet hits 0, it is substantially wiped out by the dissipative δ-function. A reasonable qualitative proposal is that the 'special states', the ticking eigenfunctions of $A_t^\dagger A_t$, are a sequence of packets that strike $\phi = 0$ at time intervals of about 2π. This argument also suggests that the best state for ticking is $D(\phi)$ itself.

The foregoing reasoning does not tell the whole story, but its substantial correctness can be seen by watching the time evolution of the norm of a state that begins as a Dirichlet kernel centred at $-2\pi/5$, see figure 3. The actual eigenfunctions of $A_t^\dagger A_t$ drop more completely than do the localized packets described above; moreover, as t is varied, the best initial function, even within one tick, changes. Evolution of these functions reveals qualitative similarity to the simple localized packets, but the actual special states have coherences beyond localization in ϕ-space.

We next address the optimization of α. For $N = 2$ the eigenvalues of $A_t^\dagger A_t$ are ($Q = \mathrm{i}\sigma_z/2 + \alpha\eta\eta^\dagger$, $\eta^\dagger = (1,1)/\sqrt{2}$)

$$\lambda_\pm = e^{-\alpha t}\left[u^2 + (\alpha^2+1)v^2 \pm 2\alpha v\sqrt{u^2+v^2}\right]$$

with $u = \cosh(\frac{1}{2}t\sqrt{\alpha^2-1})$ and $v = \left[\sinh(\frac{1}{2}t\sqrt{\alpha^2-1})\right]/\sqrt{\alpha^2-1}$. At $\alpha = 1$ (in $N = 2$), the solutions go from distinct asymptotic exponentials to oscillations about a single exponential. Note that for all α, $\lambda_+ = 1 - \mathrm{O}(t^2)$. (Initial decay *can* be linear in t, because replacing $\mathrm{Tr}\,\mathrm{e}^{-\mathrm{i}H_M t}$ by a δ-function gave the necessary frequency components.) For large N, we use Green function and Laplace transform techniques on equation (5). Letting $\hat{X}(\sigma)$ (an N-vector) be the Laplace transform of $X(t)$, we have

$$\langle\eta|\hat{X}(\sigma)\rangle = \left\langle\eta\left|\frac{1}{\sigma+\mathrm{i}H_N}\right|X_0\right\rangle(1+\alpha\Delta(\sigma))^{-1}$$

with $\Delta(\sigma) = \langle\eta|(\sigma+\mathrm{i}H_N)^{-1}|\eta\rangle$. We are thus led to study the roots of $1+\alpha\Delta(\sigma)$. As α goes from large to small, the roots undergo an interesting traverse in the complex plane starting (with one exception) and ending on the imaginary σ-axis—consistent with small decay for both large and small α. For large α there is one root on the far left; the others are nearly pure imaginary. As α decreases the far left root moves rapidly toward the others which in turn move slightly left. For even N, there is a second real root that moves (from 0) to meet the incoming large, negative root. The other roots also move left, but not as far. The roots reach their leftmost extremes at values of α that approximately coincide with the meeting of the real roots. The latter bounce off each other and now all roots move toward the imaginary axis. We have shown that the large N limit of the α value for which the real roots meet is $1/\pi$. For odd N, a variant of this scenario also selects $\alpha = 1/\pi$. When the spectrum is not regular (as for the Wigner surmise) the orchestrated bounce does not take place, but the overall picture is the same.

This behaviour of the roots is correlated with the ticking. For large α essentially only one tick exists; further decay is suppressed. The 'best' ticking occurs at the maximal leftward excursion of all the roots.

The phenomenon we have described was discovered while looking for 'special' measurement states in quantum mechanics, states whose existence is predicted by the measurement theory of [2]. That theory postulates particular initial states of a measurement apparatus that go entirely to one or another of the possible final (and macroscopically distinguishable) outcomes of a measurement. This is in contrast to the result of *typical* initial conditions for which one obtains a superposition of several macroscopically distinct states. Dealing with that superposition is the problem of the quantum theory of measurement. According to [2], one requires neither 'collapse' nor 'many-worlds' (or perhaps 'many-points-of-view' [7]) but rather that the initial state of the system preclude the occurrence of such superpositions.

Our current results argue in favour of the existence of the states postulated in [2]. In the present work we have a linear system whose average behaviour is the gradual decay out of a subspace $\mathcal{H}_N$. Nevertheless, we find there to be special initial conditions such that a state with such an initial condition is undecayed for an extended interval, whereupon it rapidly and decisively decays. One thus has collapse-like behaviour in a linear theory. In other words, at any particular time there is a subspace of (nearly) fully decayed and of (nearly) fully undecayed states. A stronger property, not required by [2], is that these subspaces substantially exhaust $\mathcal{H}$.

For the system of the present paper, the probability that a random initial state has decayed equals the ratio of the dimension of the fully decayed subspace to the sum of the dimensions of all special state subspaces. That this should be the case was also suggested in [2] and is the basis for the recovery of the usual probability predictions of quantum mechanics.

Finally, we report briefly on our efforts to determine the extent to which ticking occurs in Nature, either occasionally or generically. A single mode decaying to a continuum does not show ticking. N such levels, interacting through the same coupling that causes them to decay, provide a model in which the common origin of the mutual and decay coupling can account for the parameter α falling in the ticking range.

Alternatively, consider a single mode decaying to a continuum in the presence of peripheral degrees of freedom that are ordinarily ignored. Suppose the latter have the property that the 'phonon' emitted in the decay causes transitions among them. A possible Hamiltonian is $H_1 + H_2$ with

$$\begin{aligned} H_1 &= U\sigma_z + \sum_{k=1}^{M} \omega_k b_k^\dagger b_k + \frac{1}{\sqrt{M}} \sum_{k=1}^{M} \left[c_k \sigma_+ b_k + \text{ adjoint}\right] \\ H_2 &= \sum_{q=1}^{N} E_q a_q^\dagger a_q + \left(\sum b_k^\dagger b_k\right) \sum \left(\beta a_{q+1}^\dagger a_q + \text{ adjoint}\right) \end{aligned} \tag{9}$$

where $2U > 0$ is the decay energy, b_k boson field operators and a_k can be either boson or fermion operators. Remarkably, we have found that this system also exhibits ticking. However, computational limitations have prevented a full-scale exploration of this case.

Acknowledgments

We thank M Sassetti and U Weiss for helpful discussions. One of us (LSS) wishes to thank the Universities of Paris and Stuttgart for their kind hospitality. This work was supported in part by NSF grants PHY 88-11106, PHY 89-58506, PHY 89-07755 and PHY 90-15858. Much of the numerical work was performed with the MATLAB software package, provided (to CRD) by The Math Works, Inc.

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